Efficient valuation of American floating strike lookback options

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Abstract

Through a change of numeraire, the valuation problem for floating strike lookback options can be reduced to one dependent on a single state variable. Under this new measure, we develop an efficient method to compute the values and early exercise boundaries of American floating strike lookback options. A key idea underlying the method is the reduction of option valuation to a single optimal stopping problem for reflecting Brownian motion, indexed by one parameter in the absence of dividends and by two parameters in the presence of a dividend rate. Numerical results obtained by this method show that, after a space-time transformation, the stopping boundaries are well approximated by certain linear splines with a few knots, leading to fast and accurate approximations for American floating strike lookback option values using a decomposition formula for the American values.

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1 Introduction

A complete solution of an American option problem should provide both the option value and an optimal exercise strategy. As a result of this additional complexity in the problem, analytic solutions are not often available. In the case of American floating strike lookback options, numerical procedures proposed in the literature have been based on extending the binomial tree approach of Cox et al. (1979). Working with an augmented state process comprising the underlying geometric Brownian motion price and its running extremum (minimum for a call or maximum for a put), Barraquand and Pudet (1996) introduced a forward shooting grid method to solve this two-dimensional pricing problem. A crucial simplification is, however, possible because the pricing problem is homogeneous in the price of the underlying security. As a result, the value function of American floating strike lookback options can be expressed in terms of a single state variable after a change of numeraire from cash to the underlying security. This change of numeraire enabled Babbs (2000) to develop a binomial approximation scheme in a single new variable (ratio of the running extremum to the price of the underlying security) for the valuation of these options by incorporating a reflecting barrier at zero. This leads to substantial improvements in computational efficiency (over the forward shooting grid method).

In part due to its dependence on a prescribed root node, the procedure of Babbs (2000) computes the value (but not the early exercise boundary)
of an American floating strike lookback option. In this paper, we propose a new approach to compute both the optimal exercise boundary and the prices of these options, with common interest and dividend rates but possibly different maturities, after the same change of numeraire used in Babbs (2000). We make use of a space-time transformation that reduces the valuation of a family of American floating strike lookback options to a single canonical optimal stopping problem for reflecting Brownian motion, indexed by one (resp. two) parameter(s) in the absence (resp. presence) of dividends. Because this transformation removes the dependence on a prescribed root node when an approximating random walk is used in place of reflecting Brownian motion, we are able to compute not only the option price but also the entire early exercise boundary by backward induction. Whereas Babbs (2000) used a binomial lattice that matches the first two moments of the new state variable, our random walk approximation directly matches the simpler first two moments of reflecting Brownian motion. Further, as calendar time is scaled by $\sigma^2$ (where $\sigma$ denotes the volatility) when transformed into “canonical” time, it follows that the canonical time horizon $\sigma^2 T$ is only a small fraction of $T$ for values of $\sigma$ (between 0.1 and 0.4) that commonly arise. Consequently, a single algorithm can be implemented to solve for the early exercise boundary of all options (with any expiration date) sharing the same values of the other one or two parameters.

We also derive analytic approximations to further reduce the computational task. A decomposition formula for an American floating strike lookback option
as the sum of the corresponding European value and an “early exercise premium” is used to develop fast approximate methods to compute option prices, similar to those developed by Ju (1998) and Ait-Sahalia and Lai (1999) for American vanilla options, Ait-Sahalia et al. (2003) for American barrier options, and Lai and Lim (2004) for American fixed strike lookback options. Specifically, extensive computations of early exercise boundaries using the random walk approach show that these boundaries in the (reflecting) Brownian motion coordinates are well approximated by linear splines with a few knots, which in turn provide fast and accurate closed-form approximations to the early exercise premium. Since there is only a single state variable after the change of numeraire, these approximate American floating strike lookback prices are as easy to compute as their vanilla counterparts in Ju (1998) and Ait-Sahalia and Lai (1999).

Our method is applicable to more general “fractional” lookback options where the strike is fixed at some fraction over (for a call) or below (for a put) the realized extremum, of which the “standard” lookback options considered by Barraquand and Pudet (1996) and Babbs (2000) are special cases. This generalization makes it possible for floating strike lookback options to become out of the money (rather than always being in or at the money), thereby addressing the issue of standard lookback options being exceedingly expensive. We therefore develop our approach in the more general setting of American lookback options with fractional floating strikes.
The paper is organized as follows. Section 2 derives the pricing problem for American floating strike lookback options after a change of numeraire and a decomposition formula for an American floating strike lookback option as the sum of the corresponding European value and an early exercise premium. Results for perpetual lookback options and for finite-horizon options near expiration are also derived and used to provide bounds on the optimal stopping boundary. In Section 3, we describe the change of variables leading to a canonical optimal stopping problem and the Bernoulli random walk approach to compute the optimal stopping boundary. Numerical results obtained by applying this approach to both American floating strike lookback calls and puts are presented. In Section 4, we obtain a closed-form expression for the early exercise premium by applying a linear spline approximation to the early exercise boundary, thereby leading to two fast approximate methods for computing the option prices. One method is based on a tabulation approach while the other is based on an integral equation defining the early exercise boundary. The accuracy of our approximate option values is evaluated in a numerical comparison with benchmark values. Section 5 summarizes and concludes the paper.
2 Change of numeraire and a decomposition formula

In the standard Black-Scholes environment, the price of a security (under the risk-neutral measure $\tilde{Q}$) is represented by a geometric Brownian motion

$$S_t = S_0 \exp\{(r - q - \sigma^2/2)t + \sigma \tilde{B}_t\},$$

where $S_0$ is the initial security price and $\{\tilde{B}_t, \ t \geq 0\}$ is a standard Brownian motion with $\tilde{B}_0 = 0$. Here, $r$ is the riskless rate of return, $q$ stands for the dividend rate paid by the underlying security and $\sigma$ is the standard deviation of the security’s return (i.e., the volatility). It follows from Theorem 5.4 of Karatzas (1988) that in the absence of arbitrage opportunities the price $\tilde{P}(t, S_t, S^*_t)$ of an American fractional lookback option at time $t \in [0, T]$ before exercise is given by

$$\tilde{P}(t, S, S^*) = \sup_{\tau \in T_{a,b}} \tilde{E}\left\{e^{-r(t-\tau)} \tilde{f}(S_{\tau}, S^*_\tau) \mid S_t = S, S^*_t = S^*\right\}, \quad (1)$$

where $T_{a,b}$ is the set of stopping times taking values between $a$ and $b$ ($> a$). The payoff function is given by $\tilde{f}(S, S^*) = \max\{\epsilon(S - \theta S^*), 0\}$ with $\epsilon = +1, \theta \geq 1$, $S^*_\tau = \min_{u \in [0, \tau]} S_u$ for the call, and $\epsilon = -1, 0 < \theta \leq 1$, $S^*_\tau = \max_{u \in [0, \tau]} S_u$ for the put. We shall call $\theta$ the “moneyness” coefficient. Note that $\theta = 1$ for standard lookback options.

A change of numeraire from cash to the security is formally a change of measure from $\tilde{Q}$ to $Q$ with $d\tilde{Q}/d\tilde{Q} = \xi_T$, where $\xi_t = e^{-(r-q)t} S_t/S_0$. Under $Q$,

$$S_t = S_0 \exp\{(r - q + \sigma^2/2)t + \sigma B_t\}, \quad (2)$$
where \( \{B_t, \ t \geq 0\} \) is a standard Brownian motion given by \( B_t = \tilde{B}_t - \sigma t \).

Applying rules of conditional expectation and invoking the optional sampling theorem (cf. Karatzas and Shreve, 1991, p. 19), we can rewrite (1) in the form
\[
\tilde{P}(t, S, S^*) = SP(t, R), \quad \text{where} \quad R = S^*/S
\]
and
\[
P(t, R) = \sup_{\tau \in \mathcal{T}_{t, \tau}} \mathbb{E}\left\{e^{-q(\tau-t)} f(R_\tau) \mid R_t = R\right\}. \tag{3}
\]

Here, the expectation is taken with respect to the new measure \( \mathbb{Q} \) and \( f(R) = \max\{\epsilon(1 - \theta R), 0\} \). The single new state variable \( R_t \) is the ratio of the running extremum (minimum for a call and maximum for a put) to the price of the underlying security and has the following distribution:

**Proposition 1** The conditional distribution of \( R_t = S^*_t/S_t \) is given by
\[
\Pr(\epsilon R_\tau \leq \epsilon w \mid R_t = R) = F_\epsilon(R, w, \tau - t), \tag{4}
\]
with \( 0 < w \leq 1 \) if \( \epsilon = +1 \) (\( S^* = \min \)) and \( w \geq 1 \) if \( \epsilon = -1 \) (\( S^* = \max \)), where
\[
F_\epsilon(v, w, \tau) = N(-\epsilon(d(v, \tau) - (\sigma \sqrt{\tau})^{-1} \ln w)) + w^{2\mu} N(\epsilon(d(v, \tau) + (\sigma \sqrt{\tau})^{-1} \ln w)), \tag{5}
\]
\( N(\cdot) \) denotes the standard normal distribution function, and \( d(v, \tau) = (\sigma \sqrt{\tau})^{-1} \ln v + \mu \sigma \sqrt{\tau} \).

**Proof.** See Appendix.

Straightforward modifications of the arguments in Jaillet et al. (1990, Propositions 2.2 and 3.5) show that \( P(t, R) \) is continuous, and that \( P(t, \cdot) \) is nonin-
creasing for the call and nondecreasing for the put. Thus, the optimal exercise boundary \( \overline{R}(t) \) can be defined as the largest (resp. smallest) \( R \) such that \( P(t, R) = f(R) \) for the call (resp. put). It is then optimal to exercise the call (resp. put) option at the first time when \( R \) goes below (resp. above) \( \overline{R}(t) \).

The American floating strike lookback option problem is therefore reduced to solving for the new value function \( P(t, R) \) given by (3) and the associated boundary \( \overline{R}(t) \).

As shown by Wilmott et al. (1993, pp. 214–215), \( P \) satisfies the following partial differential equation in the continuation region \( \{(t, R) : \epsilon R > \epsilon \overline{R}(t)\} \):

\[
\mathcal{L} P := -qP + \frac{\partial P}{\partial t} + (q - r)R \frac{\partial P}{\partial R} + \frac{1}{2} \sigma^2 R^2 \frac{\partial^2 P}{\partial R^2} = 0, \tag{6}
\]

subject to

\[
\lim_{\epsilon R \downarrow \epsilon \overline{R}(t)} P(t, R) = \epsilon(1 - \theta R) \quad \text{and} \quad \lim_{\epsilon R \downarrow \epsilon \overline{R}(t)} \frac{\partial P(t, R)}{\partial R} = -\epsilon \theta. \tag{7}
\]

The second condition in (7) corresponds to the condition of smooth fit at the exercise boundary, which implies that \( P \) is continuously differentiable at all \( (t, R) \) with \( 0 \leq t < T \). Moreover, \( P \) is twice continuously differentiable in \( R \) except along the curve \( R = \overline{R}(t) \). This allows us to express an American floating strike lookback option as the sum of a corresponding European lookback option and an early exercise premium. For the rest of the paper, we let

\[
\rho = \frac{r}{\sigma^2}, \quad \gamma = \frac{q}{r}, \quad \mu = \gamma \rho - \rho - 1/2. \tag{8}
\]

**Proposition 2** After a change of numeraire, the value of an American float-
ing strike lookback option (in units of the underlying security) can be expressed as

\[ P(t, R) = P_E(t, R; T, \theta) + r \int_t^T e^{-q(\tau-t)} \epsilon[\gamma - \theta \overline{R}(\tau)]F_v(R, \overline{R}(\tau), \tau-t) \, d\tau \]

\[ + r \int_t^T \theta \overline{R}(\tau)P_E(t, R; \tau, \overline{R}(\tau)^{-1}) \, d\tau, \]

(9)

where \( P_E(t, R; T, \theta) \) denotes the European value with expiration \( T \) and moneyness coefficient \( \theta \), \( \overline{R}(\tau) \) is the optimal stopping boundary at time \( \tau \), and \( F_v(v, w, \tau) \) is given by (5). If \( \gamma \neq 1 \), the value of the European floating strike lookback option is given by

\[ P_E(t, R) = \epsilon \theta e^{-\gamma \tau} \left[ \frac{R^{2\rho(1-\gamma)}}{2 \rho(1-\gamma)} N(\epsilon(\lambda_3 - \kappa)) - RN(-\epsilon(\lambda_2 + \kappa)) \right] \]

\[ + \epsilon e^{-q \tau} \left[ N(-\epsilon(\lambda_1 + \kappa)) - \frac{\theta^{2\rho(1-\gamma)+1}}{2 \rho(1-\gamma)} N(\epsilon(\lambda_1 - \kappa)) \right], \]

(10)

with \( \tau = T - t \), \( \kappa = (\sigma \sqrt{\tau})^{-1} \ln \theta \), \( \lambda_1 = (\sigma \sqrt{\tau})^{-1} \ln R + \mu \sigma \sqrt{\tau} \), \( \lambda_2 = \lambda_1 + \sigma \sqrt{\tau} \), and \( \lambda_3 = \lambda_1 + 2 \rho(1-\gamma) \sigma \sqrt{\tau} \); for \( \gamma = 1 \),

\[ P_E(t, R) = \epsilon e^{-\tau r} [N(-\epsilon(\lambda_1 + \kappa)) - \theta RN(-\epsilon(\lambda_2 + \kappa))] \]

\[ + \theta e^{-\tau r} \sigma \sqrt{\tau} [\epsilon(\lambda_3 - \kappa)N(\epsilon(\lambda_3 - \kappa)) + n(\epsilon(\lambda_3 - \kappa))], \]

(11)

where \( n(z) = e^{-z^2/2} / \sqrt{2\pi} \) is the standard normal density.

**Proof.** See Appendix.

Since \( P(t, \overline{R}(t)) = \epsilon(1 - \theta \overline{R}(t)) \), Proposition 2 yields the following integral
equation for the early exercise boundary $\mathbf{R}(t)$:

$$
\epsilon(1 - \theta \mathbf{R}(t)) = P_E(t, \mathbf{R}(t); T, \theta) \\
+ r \int_t^T e^{-q(\tau-t)} \epsilon[\gamma - \theta \mathbf{R}(\tau)] F_e(\mathbf{R}(t), \mathbf{R}(\tau), \tau - t) \, d\tau \\
+ r \int_t^T \theta \mathbf{R}(\tau) P_E(t, \mathbf{R}(t); \tau, \mathbf{R}(\tau)^{-1}) \, d\tau.
$$

(12)

In the Appendix we make use of this integral equation to derive the following results on $\mathbf{R}(\cdot)$.

**Proposition 3** (a) For perpetual lookback options (corresponding to $T = \infty$) with $q > 0$, we have $\mathbf{R}(t) \equiv \mathbf{R}$ for all $t$, where $\mathbf{R}$ is the unique solution, in $(0, \theta^{-1})$ for the call and in $(\theta^{-1}, \infty)$ for the put, of the following equation in $x$ with $\alpha_1 = -\mu + (\mu^2 + 2\gamma \rho)^{1/2}$ and $\alpha_2 = -\mu - (\mu^2 + 2\gamma \rho)^{1/2}$:

$$
\theta \alpha_2 (1 - \alpha_1) x^{\alpha_1 - \alpha_2 + 1} + \alpha_1 \alpha_2 x^{\alpha_1 - \alpha_2} - \theta \alpha_1 (1 - \alpha_2) x - \alpha_1 \alpha_2 = 0.
$$

(13)

When $q = 0$, $\mathbf{R} = 0$ for the call and $\mathbf{R} = \infty$ for the put.

(b) For finite-horizon lookback options, we have $\mathbf{R}(t) \to \mathbf{R}(T)$ as $t \to T$, where

$$
\mathbf{R}(T) := \begin{cases} 
\min\{\gamma, 1\} / \theta & \text{for the call}, \\
\max\{\gamma, 1\} / \theta & \text{for the put}.
\end{cases}
$$


3 A canonical optimal stopping problem and its numerical solution using Bernoulli walks

We can reduce the number of parameters \( r, q, \sigma, T \) in the optimal stopping problem (3) by introducing the following change of variable \( s \):

\[
    s = \sigma^2 (t - T), \quad z = \ln R, \quad p(s, z) = e^{q(T-t)} P(t, R). \tag{14}
\]

The Markovian structure of the problem and the scaling property of Brownian motion lead to the following canonical optimal stopping problem for American floating strike lookback options, with \( \epsilon = +1, -1 \) for the call and put, respectively.

**Proposition 4** The value function of an American floating strike lookback option is

\[
p(s, z) = \sup_{\tau \in T_{s,0}} \mathbb{E}\{g(\tau, Z_\tau(s, z))\} \quad \text{with} \quad s \in [-\sigma^2 T, 0], \tag{15}
\]

where \( g(u, w) = e^{-\gamma u} \max\{\epsilon (1 - \theta e^w), 0\}, W_u^{(\mu)} = \mu u - W_u, \) and

\[
    Z_u(s, z) = W_u^{(\mu)} - \epsilon \max \left\{ 0, \max_{s \leq t \leq u} \epsilon W_t^{(\mu)} \right\} \quad \text{with} \quad W_s^{(\mu)} = z. \tag{16}
\]

Note that \( \{W_u^{(\mu)}, u \leq 0\} \) is a Brownian motion with drift \( \mu \). The value function \( p(s, z) \) given by (15) is defined on \( \mathcal{A} = (-\infty, 0] \times (-\infty, 0] \) for the call, and on \( \mathcal{A} = (-\infty, 0] \times [0, \infty) \) for the put. From its definition in (14), \( p(s, z) \) is continuous and nonincreasing (resp. nondecreasing) in \( z \) for the call (resp. put). Thus, the optimal stopping boundary \( \tau(s), s \leq 0 \), is the largest
$z \leq 0$ (resp. smallest $z \geq 0$) such that $p(s, z) = g(s, z)$. The stopping region $\mathcal{S}$ is the collection of all points in $\mathcal{A}$ such that $\epsilon z \leq \epsilon \overline{f}(s)$ and the continuation region is $\mathcal{A} \setminus \mathcal{S}$. We can retrieve the solution $(P, \overline{R})$ to the original pricing problem (3) by the transformations $P(t, R) = e^{\gamma ps} p(s, z)$ and $\overline{R}(t) = e^{\overline{f}(s)}$.

Since $g(s, z)$ is also a continuous function, it follows that $\overline{f}(s)$ is continuous. Moreover, by applying backward induction to the corresponding discrete-time optimal stopping problem for a normal random walk with increments having mean $\mu \delta$ and variance $\delta$ (with time restricted to the set $\{0, -\delta, -2\delta, \ldots\}$) and taking the limit as $\delta \to 0$, it can be shown for the call (resp. put) that $\overline{f}(s)$ is nonincreasing (resp. nondecreasing). By making use of Proposition 3 and the monotonicity of $\overline{f}(s)$ with $\overline{f}(-\infty) = \ln \overline{R}$ and $\overline{f}(0) = \ln \overline{R}(T)$, we obtain the following proposition which shows that American floating strike lookback calls written on a non-dividend paying security are exercised only at maturity, since $\overline{f}(-\infty) = \overline{f}(0) = -\infty$ when $\gamma = 0$; see Conze and Viswanathan (1991, equations (24) and (28)).

**Proposition 5** For all $s \leq 0$, the stopping boundary $\overline{f}(s)$ for the canonical optimal stopping problem (15)–(16) satisfies

$$
\overline{f}(-\infty) \leq \overline{f}(s) \leq \overline{f}(0) := \min\{\ln \gamma, 0\} - \ln \theta \quad \text{for the call},
$$

$$
\max\{\ln \gamma, 0\} - \ln \theta =: \overline{f}(0) \leq \overline{f}(s) \leq \overline{f}(-\infty) \quad \text{for the put},
$$

where $\overline{f}(-\infty) = \ln \overline{R}$ and $\overline{R}$ is uniquely determined by equation (13).
The change of variables (14) transforms (3) into the optimal stopping problem (15) for Brownian motion with drift, subject to a reflecting barrier at 0 as given in (16). As the horizon for problem (15) is always 0, only one numerical program need be implemented for all expiration dates \( T \) sharing a given set of parameters \((\rho, \gamma, \theta)\). Also, from (14), \( s = -\sigma^2 T \) at time \( t = 0 \) and \( \sigma^2 T \) is typically small (not exceeding 0.3) for volatilities \((0.1 \leq \sigma \leq 0.4)\) and expirations \((0.08 \leq T \leq 1.5)\) that commonly arise in practice. In view of the functional central limit theorem, we can approximate Brownian motion with drift \( \mu \) by an asymmetric Bernoulli random walk with time increment \( \delta > 0 \) and space increments of magnitude \( h = \sqrt{\delta (1 + \delta \mu^2)^{1/2}} \), and compute \( p(s, z) \) via the following backward recursion algorithm applied to the approximating random walk. Specifically, with \( s_0 = 0 \), \( s_j = s_{j-1} - \delta \) for \( j \geq 1 \), and \( z \in \mathbb{Z}_h = \{-\epsilon nh : n = 0, 1, 2, \ldots\} \), the value function (15) can be approximated by the backward induction

\[
p(s, z) = \max \left\{ g(s, z), \pi_+ p(s_{i-1}, z_+) + \pi_- p(s_{i-1}, z_-) \right\},
\]

where \( \pi_{\pm} = (1 \pm \mu \delta / h) / 2 \), \( z_+ = (z + h)I_{\{z \leq -h\}} + (-h)I_{\{z = 0\}} \) and \( z_- = z - h \) for the call option, and \( z_+ = z + h \) and \( z_- = (z - h)I_{\{z \geq h\}} + hI_{\{z = 0\}} \) for the put option. Note that in contrast to the lattice scheme of Babbs (2000), we always have a layer of nodes lined up at the reflecting barrier for all choices of \( \delta \). Our backward induction algorithm therefore uses fewer nodes because the “reflected” nodes coincide with the “usual” nodes.
Each point \( z \in \mathbf{Z}_h \) can be determined to be a stopping or continuation point at time \( s_i \) according to whether \( p(s_i, z) = g(s_i, z) \) or \( p(s_i, z) > g(s_i, z) \). We use a “continuity correction,” proposed by Chernoff and Petkau (1986), to obtain the optimal stopping boundary for Brownian motion from the discrete-time and discrete-state boundary associated with (17). Denote by \( z_h(s_i) \) the last (i.e., largest for a call and smallest for a put) point in \( \mathbf{Z}_h \) for which \( p(s_i, z) = g(s_i, z) \). The early exercise boundary at \( s_i \) can be computed via the Chernoff-Petkau continuity correction as follows:

\[
\tau(s_i) = z_h^1(s_i) - \epsilon h |D_2(s_i, z)|/\{2D_2(s_i, z) - 4D_1(s_i, z)\},
\]

where \( z_h^j(s) = z_h(s) + \epsilon j h \) and \( D_j(s, z) = g(s, z) - p(s, z_h^j(s)) \) for \( j = 1, 2 \). Under certain conditions the continuity correction can be shown to approximate the continuous-time boundary with \( o(\sqrt{\delta}) \) error; see Lai et al. (2004). One such condition is boundedness of the derivative \( \partial \tau / \partial s \) in some neighborhood of \( s_i \). Numerical results show that the derivative is large only in a very small neighborhood of \( s = 0 \), so the Chernoff-Petkau correction can be applied when \( s_i \leq -0.005 \). For \( 0 > s > -0.005 \), \( \tau(s) \) is close to \( \tau(0) \) and the uncorrected \( z_h \) typically suffices to approximate this small portion of the early exercise boundary.

As an illustration, the backward induction (17) with \( \delta = 10^{-4} \) is implemented on American floating strike lookback options with \( \rho = 0.1 \) and various values of \( \gamma \). The optimal stopping boundaries for calls with \( \theta = 1.0, 1.2 \) and puts with \( \theta = 1.0, 0.8 \) are shown in Figure 1. Since these boundaries are rep-
resentative of lookback options with other parameter values, we see that the optimal stopping boundary in the canonical scale can be well approximated by linear splines with a few knots. We tabulate such approximations in Table 1, using knots at eight canonical time points \(s = -0.3, -0.2, -0.15, -0.1, -0.05, -0.025, -0.01, -0.005\).

4 Two fast approximate valuation methods

The decomposition formula (9) expresses an American floating strike lookback option as the sum of the corresponding European value and an early exercise premium. We now use (9) to develop fast and accurate valuation methods, both based on the linear spline approximation of the early exercise boundary suggested at the end of Section 3.

4.1 Computation of the early exercise premium

A first step in dealing with the integrals in (9) is to use the change of variables (14). To fix ideas, consider options with \(\gamma \neq 1\). With \(u = \sigma^2(\tau - T)\),

\[
h_{\epsilon, \mu}(x, y, u) = N(e((y - x)/\sqrt{u} + \mu\sqrt{u})), \quad A_{\rho, \mu}(s, z) = B_{\rho, \mu, 0}(s, z), \quad \text{and}
\]

\[
B_{\rho, \mu, \eta}(s, z) = \rho e^{\rho s} \int_s^0 e^{-\rho u + \eta \overline{z}(u)} h_{\epsilon, \mu}(z, \overline{z}(u), u - s) \, du,
\]
the early exercise premium (in units of the underlying security) is given by

\[ \Pi(s, z) = \epsilon A_{\gamma, -\mu}(s, z) - \epsilon \theta e^s A_{\rho, -\mu - 1}(s, z) - \frac{\epsilon \theta e^{-2(\mu + 1)z}}{2\mu + 1} A_{\rho, -\mu - 1}(s, -z) + \epsilon B_{\gamma, \rho, \mu}(s, -z) - \frac{\epsilon \theta}{\gamma} e^{\frac{\mu}{2\mu + 1}} B_{\gamma, \rho, 2\mu}(s, -z). \]  

(18)

At the end of Section 3, the early exercise boundary \( \overline{z}(u) \) has been computed on the grid of points \( s = u_m < \cdots < u_0 = 0 \). Since \( \int_s^0 = \sum_{i=1}^m \int_{u_i}^{u_{i-1}} \), we consider the integrals \( B_{\rho, \mu, \eta}^i(z) := \rho e^{\rho s} \int_{u_i}^{u_{i-1}} e^{-\rho u + \eta \overline{z}(u)} h_{\epsilon, \mu}(z, \overline{z}(u), u - s) \, du \) and \( A_{\rho, \mu}^i(z) = B_{\rho, \mu, 0}^i(z) \). To evaluate these integrals, we apply the following linear interpolation:

\[ \overline{z}(u) = \beta_i u + \alpha_i \quad \text{for} \quad u_i \leq u \leq u_{i-1}, \]  

(19)

where \( \beta_i = (\overline{z}_{i-1} - \overline{z}_i)/(u_{i-1} - u_i) \) and \( \alpha_i = (u_{i-1} \overline{z}_i - u_i \overline{z}_{i-1})/(u_{i-1} - u_i) \), with \( \overline{z}_i = \overline{z}(u_i) \). Then we obtain the following closed-form formulas (cf. Ju, 1998; AitSahlia and Lai, 1999):

\[ A_{\rho, \mu}^i(z) = e^{-\rho \tau_i} H(b_i, c_i, \tau_i) - e^{-\rho \tau_{i-1}} H(b_i, c_i, \tau_{i-1}) + e^{(a_i - b_i) c_i} \left( b_i/a_i + 1 \right) \left[ H(a_i, c_i, \tau_{i-1}) - H(a_i, c_i, \tau_i) \right]/2 + e^{-(a_i + b_i) c_i} \left( b_i/a_i - 1 \right) \left[ H(a_i, -c_i, \tau_{i-1}) - H(a_i, -c_i, \tau_i) \right]/2 \]

and \( B_{\rho, \mu, \eta}^i(z) = (\rho - \eta \beta_i)^{-1} \rho e^{\eta (a_i + \beta_i u_m)} A_{\rho - \eta \beta_i, \mu}^i(z) \), where \( \tau_i = u_i - u_m, \quad a_i = (b_i^2 + 2\rho)^{1/2}, \quad b_i = \epsilon (\mu + \beta_i), \quad c_i = c_i(z) = \epsilon (\beta_i u_m + \alpha_i - z), \) and \( H(b, c, \tau) = N(b\tau^{1/2} + c\tau^{-1/2}) \).
Collecting terms, the early exercise premium for $\gamma \neq 1$ is approximated by

$$
\Pi(s, z) \approx \epsilon \sum_{i=1}^{m} A_{\gamma \rho - \mu}^i(z) - \epsilon \theta \sum_{i=1}^{m} \left[ e^{\frac{\epsilon}{\gamma} A_{\gamma \rho - \mu - 1}^i(z)} + \frac{e^{-(2\mu+1)z}}{2\mu + 1} A_{\gamma \rho - \mu - 1}^i(-z) \right] 
+ \epsilon \sum_{i=1}^{m} B_{\gamma \rho, \mu, 2\mu}^i(-z) - \frac{\epsilon \theta}{\gamma} \frac{2\mu}{2\mu + 1} \sum_{i=1}^{m} B_{\gamma \rho, \mu, 2\mu + 1}^i(-z),
$$

where $\mu = \gamma \rho - \rho - 1/2$ as before. A similar expression can be obtained for the early exercise premium when $\gamma = 1$.

4.2 Two approximations to the optimal stopping boundary

A simple approximation to $\tau(\cdot)$ is obtained by the linear interpolation (19) of tabulated values of $\tau_i$ (such as those given in Table 1). The early exercise premium (18) can then be evaluated by (20).

Instead of relying on tabulated values of $\tau(u_i)$ that have been computed previously with high precision by the Bernoulli walk method, we can generate them by solving recursively a system of nonlinear equations. Recasting the integral equation (12) in the canonical variables (14), we can solve for the boundary recursively over a grid of points $0 = s_0 > s_1 > \cdots > s_m = -\sigma^2 T$.

The recursive procedure is initialized at $s_0 = 0$ by $\tau_0 = \min\{\ln \gamma, 0\} - \ln \theta$ for a call and $\tau_0 = \max\{\ln \gamma, 0\} - \ln \theta$ for a put. Suppose that $\tau_0, \ldots, \tau_{k-1}$ have been determined. Then, applying linear interpolation to $\tau(u)$ as in (19), the early exercise premium $\Pi(s_k, z)$ at $s_k$ can be obtained from (20) with $m = k$. Therefore the integral equation for $\Phi(t)$ leads to the following nonlinear
equation for $\mathbf{z}_k = \mathbf{z}(s_k)$:

$$\epsilon(1 - \theta e^{z_k}) = e^{\gamma ps_k} p_E(s_k, \mathbf{z}_k) + \Pi(s_k, \mathbf{z}_k),$$

(21)

where $p_E(s, z)$ is defined by applying the change of variables (14) to $P_E(t, R)$ in equation (10) or (11).

We can use a bracketing technique to search for the solution $\mathbf{z}_k$. Let $D(\cdot)$ denote the difference between the two sides of (21), i.e., $D(\cdot) = \text{l.h.s.} - \text{r.h.s.}$ Since $\mathbf{z}_{k-1}$ is a continuation point at $s_k$, initializing at $\hat{z}_0 = \mathbf{z}_{k-1}$ yields $D(\hat{z}_0) < 0$. We can then increment $\hat{z}_0$ in steps of $-\epsilon \omega$ (with $\epsilon = +1, -1$ for the call and put, respectively) to search for the smallest integer $\ell_0$ such that $D(\hat{z}_0 - \ell_0 \epsilon \omega) > 0$. After bracketing the solution in this way between $\hat{z}_0$ and $\hat{z}_1 := \hat{z}_0 - \ell_0 \epsilon \omega$ with $D(\hat{z}_0) < 0$ and $D(\hat{z}_1) > 0$, we can use successive linear approximations of $D(z)$ to find the solution within some prescribed tolerance level. As pointed out in Section 3, this integral equation approach is in fact preferred to the Bernoulli walk approach for $s$ very close to $0$, because the boundary has infinite gradient as $s \to 0$ and the Chernoff-Petkau correction cannot be used there and because only a small number $m$ of time steps in (19) is needed to produce a highly accurate piecewise linear approximation of $\mathbf{z}(u)$ for $s \leq u \leq 0$ when $s$ is small.

4.3 Numerical results

We present a numerical study of the two approximations just described. Benchmark values for our numerical study are computed using the procedure
of Babbs (2000). Also working under a change of numeraire to the underlying security, a Cox-Ross-Rubinstein (CRR) binomial lattice is formed, incorporating a reflecting barrier at zero. Option valuation then proceeds by way of backward recursion over the recombinant lattice. As Broadie et al. (1999) demonstrates, the efficiency of this procedure can be enhanced by applying a two-point extrapolation. Specifically, one computes two estimates $P(n)$ and $P(2n)$, based respectively on $n$ and $2n$ time steps, and obtains an improved price estimate as $2P(2n) - P(n)$. We note two distinctions between the CRR lattice of Babbs (2000) and our Bernoulli walk approximation in Section 3. For one, the CRR lattice uses “risk neutral” probabilities while our Bernoulli walk uses probabilities induced by the underlying Brownian motion with drift. Moreover, the change of variables (14) removes the dependence on a root node as in the CRR lattice. As a result, we can obtain in a single backward induction values of lookback options with (essentially) all maturities.

In our numerical comparison, we take $S = 100$ and $r = 0.04$, and adopt a representative selection of values for the other parameters (see Table 2): $\sigma = 0.2, 0.4$, $\gamma = 0.5, 1.0, 1.5$ (so $q = 0.02, 0.04, 0.06$), $\tau = T - t = 0.5, 1.0$, $\theta = 1.0, 1.2$, and $R = 1.0, 0.7$. Our estimates using the decomposition formula are obtained by first solving for the optimal stopping boundary via backward induction (17) on a horizon of 0.3 with $\delta = 10^{-4}$ and then evaluating (20). For the integral equation approach, we solve (21) over a regular grid $s_i = -i\delta$ ($1 \leq i \leq m$) with the early exercise premium approximated by (20); for $\sigma = 0.2$ (resp. 0.4), we take $\delta = 2.5 \times 10^{-4}$ (resp. $10^{-3}$) on a horizon of 0.04...
On a notebook computer with a Pentium IV 1.8 GHz processor and 768 MB of RAM, it took 46.75 and 31.23 seconds to price the sample of 48 lookback calls listed in Table 2, using the tabulation-interpolation approach and the integral equation approach, respectively. Both sets of price estimates are always within penny accuracy relative to the benchmark values computed by Babbs’ procedure; in many cases the price estimates are indistinguishable from the benchmark values. We also report in Table 2 the early exercise boundaries $\overline{R}(t)$, computed using the Bernoulli walk approach of Section 3 and the integral equation approach of Section 4.2. We note that it is optimal to exercise lookback calls with $R < \overline{R}(t)$ (in which case $P(t, R) = 1 - \theta R$, the early exercise value); there are three such cases (boxed) in Table 2.

**5 Conclusion**

This paper describes a Bernoulli walk method to compute both the value and early exercise boundary of an American floating strike lookback option, after a change of numeraire to the underlying security. The Bernoulli walk is a natural alternative to the binomial tree after we use a change of variables to reduce the optimal stopping problem to its canonical form indexed by only one (resp. two) parameter(s) in the absence (resp. presence) of dividends. The time horizon in the canonical scale is considerably shorter than in the calendar time.
scale, thereby requiring less computational effort than corresponding methods based on the binomial tree. Numerical results obtained by the Bernoulli walk method show that the early exercise boundary is well approximated by a piecewise linear function in the canonical scale. This approximate shape and the decomposition formula (9) suggest two fast and accurate approximate methods to compute the values of American floating strike lookback options.

One method is based on tabulation of the exercise boundary at a few pre-specified knots over a set of parameter values and the closed-form expressions like (9) with (20) to compute the option values once this boundary approximation is determined. Such tabulation can be stored in a hand-held calculator which can be used to evaluate the closed-form expressions like (9) with (20). In fact, once the boundary is determined and a tabulation scheme is decided upon, several options with the same \((\rho, \gamma, \theta)\) can be priced in very little time. We note that a tabulation-interpolation approach based on option prices (cf. Joubert and Rogers, 1997, for standard American option values) would require much larger tables that have to be stored in a computer as a dictionary. In contrast, we only tabulate the boundary \(\overline{z}(s)\) at a few points \(s_i\) in the canonical coordinate system.

An alternative to tabulation is to solve, under the piecewise linear approximation to \(\overline{z}(s)\) with a few pieces, an integral equation characterizing \(\overline{z}\). This alternative method, which was first introduced by Ju (1998) for American vanilla options, is especially effective when \(\sigma^2(T - t)\) is small. The numerical
study in Section 4 shows that both approximations are sufficiently fast and accurate for pricing American floating strike lookback options.

Appendix

Proof of Proposition 1

Consider $S^* = \min$, the case of $S^* = \max$ being analogous. Let $X_\tau = \ln(S_\tau/S_0)$ and $Y_\tau = \ln(S^*_\tau/S_0)$, where $S_0$ is the initial security price. Then $X_\tau = \ln(S_\tau/S_t) + X_t$ and $Y_\tau = \ln(S^*_\tau/S_t) + X_t$. From the joint distribution of $(\ln(S_{\tau-t}/S_0), \ln(S^*_{\tau-t}/S_0))$ given by Lemma 2 of Conze and Viswanathan (1991), we deduce that for $-\infty < x < \infty$ and $y \leq \min\{x, Y_t\}$,

$$
\Pr(X_\tau \geq x, Y_\tau \geq y \mid X_t, Y_t) = F_{+1}(e^{x-y}, e^{X_t-y}, \tau - t) - e^{2\mu(X_t-y)};
$$

and that for $x \geq Y_t \neq X_t$,

$$
\Pr(X_\tau \geq x, Y_\tau = Y_t \mid X_t, Y_t) = \lim_{y \uparrow Y_t} \Pr(X_\tau \geq x, Y_\tau \geq y \mid X_t, Y_t).
$$

Applying the transformation $R_\tau = \exp(Y_\tau - X_\tau)$ yields (4).

Proof of Proposition 2

The European value given in (10) can be obtained from the dividend-free pricing formulas (22)–(23) of Conze and Viswanathan (1991) as follows: re-
place $T$ by $\tau$ and replace every occurrence of $r$ by $r - q$, but still use $e^{-r\tau}$ as the discount factor. The remaining case $\gamma = 1$ can be obtained via direct evaluation of the usual risk-neutral expectation or as the limiting case of (10) as $\gamma \to 1$. Taking the latter approach, we examine the Taylor series approximation of the first and last terms for $\gamma$ close to 1. Expanding up to $1 - \gamma$ yields

$$\frac{e^{-r\tau} R^{2\rho(1-\gamma)}}{2\rho(1-\gamma)} N_{3\epsilon} = \left[ \frac{1}{2\rho(1-\gamma)} + \ln R \right] e^{-r\tau} N_\epsilon + \frac{\epsilon e^{-r\tau} \sigma \sqrt{\tau}}{2} n_\epsilon,$$

$$\frac{e^{-q\tau} \theta^{2\rho(1-\gamma)}}{2\rho(1-\gamma)} N_{1\epsilon} = \left[ \frac{1}{2\rho(1-\gamma)} + \frac{\sigma^2 \tau}{2} + \ln \theta \right] e^{-r\tau} N_\epsilon - \frac{\epsilon e^{-r\tau} \sigma \sqrt{\tau}}{2} n_\epsilon,$$

where $N_{i\epsilon} = N(\epsilon(\lambda_i - \kappa))$ ($i = 1, 3$), $N_\epsilon = N(\epsilon(\lambda^* - \kappa))$, $n_\epsilon = n(\epsilon(\lambda^* - \kappa))$, and $\lambda^* = (\sigma \sqrt{\tau})^{-1} \ln R - \sigma \sqrt{\tau}/2$. Equation (11) follows by letting $\gamma \to 1$.

Next, we show that

$$P(t, R) = P_E(t, R; T, \theta) + r \int_t^T e^{-q(\tau-t)} \mathbb{E}\left[ \epsilon(\gamma - \theta R_\tau) I_{\{\epsilon R_\tau \leq \epsilon R(\tau)\}} \left| R_t = R \right. \right] d\tau. \quad (22)$$

In view of the discussion after (7) and the fact that

$$dR_t = (q - r)R_t dt + \sigma R_t dB_t + R_t dL_t,$$

where $L_t = \epsilon \min_{u \in [0,t]} \epsilon\{(r - q + \sigma^2/2)u + \sigma B_u\}$, an application of the gener-
alized Ito’s formula (cf. Krylov, 1980) to $e^{-qt} P$ yields

$$
e^{-q(T-t)} \max\{\epsilon(1 - \theta R_T), 0\}
$$

$$
= P(t, R_t) + \int_t^T e^{-q(\tau-t)} L_P \, d\tau + \int_t^T e^{-q(\tau-t)} R_\tau \frac{\partial P}{\partial R}(\sigma \, dB_\tau + dL_t),
$$

(23)

where $P(t, R_t)$ is the American value (in units of the underlying security) at time $t$ and $L$ is the partial differential operator defined in (6). First, since $L_t$ changes only when $R_t = 1$, we have

$$
\int_t^T e^{-q(\tau-t)} R_\tau \frac{\partial P}{\partial R} \, dL_t = \int_t^T e^{-q(\tau-t)} R_\tau \frac{\partial P}{\partial R} \bigg|_{R=1} \, dL_t = 0.
$$

Here, we have used the fact that the option value is insensitive to small changes in the extremum when a new extremum is realized, i.e., $\partial P/\partial R|_{R=1} = 0$. Second, writing $P = PI_{\{\epsilon R_t > \epsilon R(t)\}} + \epsilon(1 - \theta R_t)I_{\{\epsilon R_t \leq \epsilon R(t)\}}$, it follows from (6) that

$$
\int_t^T e^{-q(\tau-t)} L_P \, d\tau = -r \int_t^T e^{-q(\tau-t)} \epsilon(\gamma - \theta R_\tau)I_{\{\epsilon R_\tau \leq \epsilon R(\tau)\}} \, d\tau.
$$

Substituting this into (23) and taking expectations yield (22), noting that

$$
P_E(t, R_t; T, \theta) = \mathbb{E}[e^{-q(T-t)} \max\{\epsilon(1 - \theta R_T), 0\} \mid R_t]
$$

(24)

and that the integral with respect to Brownian motion is a martingale. Finally, since $\epsilon(\gamma - \theta R_\tau)I_{\{\epsilon R_\tau \leq \epsilon R(\tau)\}} = \epsilon(\gamma - \theta R(\tau))I_{\{\epsilon R_\tau \leq \epsilon R(\tau)\}} + \theta R(\tau) \max\{\epsilon(1 - R_\tau/R(\tau)), 0\}$, the integral in (22) evaluates to the corresponding expressions in (9) by using Proposition 1 and (24).
Proof of Proposition 3

(a) We consider the case $q > 0$, the result for the case $q = 0$ being given by Duffie and Harrison (1993). Since $R(t) \equiv R$, with $R \in (0, \theta^{-1})$ for the call and $R \in (\theta^{-1}, \infty)$ for the put, it follows from (12) that the (constant) boundary $R$ for perpetual lookback options satisfies the integral equation

$$
\epsilon(1 - \theta R)
= \epsilon(q - r\theta R) \int_0^\infty e^{-q\tau} F_\epsilon(R, R, \tau) \, d\tau + r\theta R \int_0^\infty P_E(0, R; \tau, R^{-1}) \, d\tau.
$$

From (5) and (10), we have

$$
F_\epsilon(R, R, \tau) = N(-\epsilon\mu\sqrt{\tau}) + R^{2\mu} N[\epsilon(2(\sigma\sqrt{\tau})^{-1} \ln R + \mu\sigma\sqrt{\tau})],
$$

$$
P_E(0, R; \tau, R^{-1}) = e^{-r\tau} \frac{R^{-2(\mu+1)}}{2\rho(1 - \gamma)} N[\epsilon(2(\sigma\sqrt{\tau})^{-1} \ln R - (\mu + 1)\sigma\sqrt{\tau})]
- \epsilon e^{-r\tau} N(-\epsilon\mu + 1\sigma\sqrt{\tau}) + \epsilon e^{-q\tau} N(-\epsilon\mu\sigma\sqrt{\tau})
- \epsilon e^{-q\tau} \frac{R^{2\mu}}{2\rho(1 - \gamma)} N[\epsilon(2(\sigma\sqrt{\tau})^{-1} \ln R + \mu\sigma\sqrt{\tau})].
$$

Thus, the terms in the r.h.s. of the integral equation consist of integrals of the form $\int_0^\infty e^{-a\tau} N(b\sqrt{\tau} + c/\sqrt{\tau}) \, d\tau$, where $a > 0$. Making use of the identity

$$
\int_0^\infty e^{-a\tau} N(b\sqrt{\tau} + c/\sqrt{\tau}) \, d\tau
= \begin{cases}
\frac{1}{a} + \frac{1}{2a} \left( \frac{b}{\sqrt{2a} + b^2} - 1 \right) e^{-c(b + \sqrt{2a + b^2})} & \text{if } c > 0,
\frac{1}{2a} \left( \frac{b}{\sqrt{2a} + b^2} + 1 \right) e^{-c(b - \sqrt{2a + b^2})} & \text{if } c \leq 0,
\end{cases}
$$

25
the integral equation simplifies to the following:

\[
\frac{\theta R^{1+2(\mu^2+2\gamma \rho)^{1/2}}}{2\rho(1-\gamma)} \left\{ \frac{\epsilon(\mu + 1)}{(\mu^2 + 2\gamma \rho)^{1/2}} - 1 - \frac{2\mu}{\gamma} \left[ \frac{\epsilon\mu}{(\mu^2 + 2\gamma \rho)^{1/2}} + 1 \right] \right\}
- \theta R^{2(\mu^2+2\gamma \rho)^{1/2}} \left[ \frac{\epsilon \mu}{(\mu^2 + 2\gamma \rho)^{1/2}} + 1 \right]
- \theta \frac{\epsilon(\mu + 1)}{(\mu^2 + 2\gamma \rho)^{1/2}} + \left[ \frac{\epsilon \mu}{(\mu^2 + 2\gamma \rho)^{1/2}} + 1 \right] = 0, \quad (25)
\]

since \(\epsilon \ln \bar{R} < 0\) for both the call (\(\epsilon = +1\) and \(\bar{R} < 1/\theta \leq 1\) with \(\theta \geq 1\)) and the put (\(\epsilon = -1\) and \(\bar{R} > 1/\theta \geq 1\) with \(0 < \theta \leq 1\)). For the call, setting \(\epsilon = +1\) and multiplying (25) through by \(\alpha_1(\mu^2 + 2\gamma \rho)^{1/2}\) yield

\[
(\alpha_2 + 2\gamma \rho)\theta \bar{R}^{1+2(\mu^2+2\gamma \rho)^{1/2}} - 2\gamma \rho \bar{R}^{2(\mu^2+2\gamma \rho)^{1/2}} - (\alpha_1 + 2\gamma \rho)\theta \bar{R} + 2\gamma \rho = 0; \quad (26)
\]

this is the same as (13) with \(\bar{R} < 1/\theta\), noting that \(\alpha_1 - \alpha_2 = 2(\mu^2 + 2\gamma \rho)^{1/2}\) and \(\alpha_1\alpha_2 = -2\gamma \rho\). For the put, setting \(\epsilon = -1\) and multiplying (25) through by \(\alpha_2(\mu^2 + 2\gamma \rho)^{1/2}\) again yield (26), which is equivalent to (13) with \(\bar{R} > 1/\theta\). Finally, (13) has a unique solution in the interval \((0, \theta^{-1})\) because its l.h.s. is strictly decreasing in that interval. For the interval \((\theta^{-1}, \infty)\), consider the transformation \(y = 1/x\).

(b) For notational simplicity, we write \(\bar{R}_t\) for \(\bar{R}(t)\) and \(\tau = T - t\), and rearrange the integral equation (12) into

\[
\theta \bar{R}_t = G_t \times \left\{ e^{-\tau} N(\epsilon(\sigma \sqrt{\tau})^{-1} \ln(\theta \bar{R}_t) + (\mu + 1)\sigma \sqrt{\tau}))
+ \int_0^\tau r e^{-ru} N(\epsilon(\sigma \sqrt{u})^{-1} \ln(\bar{R}_t/\bar{R}_{t+u}) + (\mu + 1)\sigma \sqrt{u})) \, du \right\}^{-1},
\]

(27)
where

$$G_t = \left\{ e^{-q \tau} N(\epsilon(\sigma \sqrt{\tau})^{-1} \ln(\theta \overline{R}_t) + \mu \sigma \sqrt{\tau}) + \int_0^\tau q e^{-q u} N(\epsilon((\sigma \sqrt{u})^{-1} \ln(\overline{R}_t/\overline{R}_{t+u}) + \mu \sigma \sqrt{u})) \, du \right\}$$

$$- \frac{\theta \overline{R}_t^{2\rho(1-\gamma)}}{2\rho(1-\gamma)} \left\{ e^{-r \tau} N(\epsilon((\sigma \sqrt{\tau})^{-1} \ln(\overline{R}_t/\theta) - (\mu + 1)\sigma \sqrt{\tau}) + \int_0^\tau r e^{-ru} N(\epsilon((\sigma \sqrt{u})^{-1} \ln(\overline{R}_t/\overline{R}_{t+u}) - (\mu + 1)\sigma \sqrt{u})) \, du \right\}$$

$$+ \frac{\theta^{2\rho(1-\gamma)+1}}{2\rho(1-\gamma)} \left\{ e^{-q \tau} N(\epsilon((\sigma \sqrt{\tau})^{-1} \ln(\overline{R}_t/\theta) + \mu \sigma \sqrt{\tau}) + \int_0^\tau r e^{-qu} (\theta \overline{R}_{t+u})^{2\rho+1} N(\epsilon((\sigma \sqrt{u})^{-1} \ln(\overline{R}_t\overline{R}_{t+u}) + \mu \sigma \sqrt{u})) \, du \right\}$$

$$- \int_0^\tau e^{-q u} (q - r \theta \overline{R}_{t+u}) \overline{R}_{t+u}^{2\rho} N(\epsilon((\sigma \sqrt{u})^{-1} \ln(\overline{R}_t/\overline{R}_{t+u}) + \mu \sigma \sqrt{u})) \, du.$$ 

Since $\theta \overline{R}_t \leq 1$ for the call and $\theta \overline{R}_t \geq 1$ for the put, we consider the following two possibilities as $\tau \to 0$:

$$\lim_{\tau \to 0} \epsilon(\sigma \sqrt{\tau})^{-1} \ln(\theta \overline{R}_t) = -\infty \quad \text{or} \quad -\infty < \lim_{\tau \to 0} \epsilon(\sigma \sqrt{\tau})^{-1} \ln(\theta \overline{R}_t) \leq 0.$$  

First, if $\epsilon(\sigma \sqrt{\tau})^{-1} \ln(\theta \overline{R}_t)$ decreases without limit as $\tau \to 0$, then so does $\epsilon(\sigma \sqrt{\tau})^{-1} \ln(\overline{R}_t/\theta)$. This in turn allows us to evaluate the limit of the r.h.s. of (27) using l’Hôpital’s rule to give $\lim_{\tau \to T} \theta \overline{R}_t = q/r = \gamma$. This holds when $\epsilon \gamma \leq \epsilon$, i.e., when $q \leq r$ for the call and when $q \geq r$ for the put. On the other hand, if the limit of $\epsilon(\sigma \sqrt{\tau})^{-1} \ln(\theta \overline{R}_t)$ as $\tau \to 0$ is bounded below, then the limit of the r.h.s. of (27) can be evaluated directly: $\lim_{\tau \to T} \theta \overline{R}_t = 1$. This holds when $\epsilon \gamma > \epsilon$, i.e., when $q > r$ for the call and when $q < r$ for the put.
References


Table 1
(a) Tabulation of $-\pi(s)$ for American floating strike lookback calls with $\rho = 0.1$. The boundaries are obtained by backward induction with a Bernoulli walk approximation; see the upper panels of Fig. 1.

(i) $\theta = 1.0$ (at the money)

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>2.6856 1.4343 1.2719 1.1706 1.0531 0.9434 0.9003 0.8631 0.8441 0.7985</td>
</tr>
<tr>
<td>0.45</td>
<td>2.5905 1.3931 1.2291 1.1242 0.9963 0.8489 0.7943 0.7575 0.7390 0.6931</td>
</tr>
<tr>
<td>0.5</td>
<td>2.5054 1.3527 1.1894 1.0838 0.9525 0.7804 0.7006 0.6626 0.6436 0.5978</td>
</tr>
<tr>
<td>0.55</td>
<td>2.4285 1.3228 1.1613 1.0552 0.9230 0.7377 0.6221 0.5748 0.5562 0.5108</td>
</tr>
<tr>
<td>0.6</td>
<td>2.2943 1.2679 1.1108 1.0069 0.8746 0.6843 0.5378 0.4241 0.4027 0.3567</td>
</tr>
<tr>
<td>0.7</td>
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</tr>
<tr>
<td>0.8</td>
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</tr>
<tr>
<td>0.9</td>
<td>1.9947 1.1528 1.0081 0.9128 0.7892 0.6080 0.4632 0.3182 0.2365 0.0000</td>
</tr>
<tr>
<td>1.0</td>
<td>1.8484 1.0974 0.9613 0.8704 0.7524 0.5789 0.4399 0.3009 0.2228 0.0000</td>
</tr>
<tr>
<td>1.2</td>
<td>1.6765 1.0331 0.9052 0.8216 0.7110 0.5470 0.4151 0.2840 0.2108 0.0000</td>
</tr>
</tbody>
</table>

(ii) $\theta = 1.2$ (out of the money)

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>2.8610 1.5618 1.4101 1.3186 1.2167 1.1244 1.0825 1.0448 1.0262 0.9808</td>
</tr>
<tr>
<td>0.45</td>
<td>2.7653 1.5119 1.3575 1.2625 1.1491 1.0248 0.9766 0.9403 0.9215 0.8755</td>
</tr>
<tr>
<td>0.5</td>
<td>2.6794 1.4713 1.3156 1.2191 1.0997 0.9496 0.8825 0.8444 0.8254 0.7802</td>
</tr>
<tr>
<td>0.55</td>
<td>2.6018 1.4346 1.2817 1.1835 1.0620 0.8963 0.8003 0.7575 0.7391 0.6931</td>
</tr>
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Table 1
(b) Tabulation of $\pi(s)$ for American floating strike lookback puts with $\rho = 0.1$; see the lower panels of Fig. 1.

(i) $\theta = 1.0$ (at the money)

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(ii) $\theta = 0.8$ (out of the money)

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Table 2
Valuation of American floating strike lookback calls with $S = 100$ and $r = 0.04$, using the tabulation-interpolation approach based on the decomposition formula (Dec) and the integral equation approach (Int). Benchmark values are computed using Babb's CRR lattice with a reflecting barrier (Bab). European values $P_E(t, R)$ are given for reference. Also reported are the early exercise boundaries $R(t)$ obtained by the Bernoulli walk approach (Ber) and the integral equation approach (Int).

(i) $\sigma = 0.2 \ (\rho = 1.0)$

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Fig. 1. Optimal stopping boundaries of American floating strike lookback options. For the call, $\gamma = 0.45, 0.5, 0.55, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5$ (lowest to highest) and stopping regions are below the corresponding boundaries. For the put, $\gamma = 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0, 2.25, 2.5$ (lowest to highest) and stopping regions are above the boundaries. Linear spline approximations of these boundaries are given in Table 1.