

Expected Number of Real Zeros of a Random Polynomial with Independent, Identically Distributed, Symmetric, Long-Tailed Coefficients

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Abstract. Kac showed that if the coefficients of a polynomial of degree n are iid standard normal then the expected number of real zeros is asymptotically $\frac{2}{\pi} \log n$. About 40 years ago, it was shown that if the iid coefficients have a symmetric stable distribution with characteristic function $e^{-|z|^\alpha}$, $0 < \alpha < 2$ then the expected number of real zeros is asymptotic to $c(\alpha) \log n$, where $c(\alpha)$, given, by a formula, increases to unity as $\alpha \downarrow 0$. The density of the symmetric stable with parameter α has tail probability $P(\xi > x) \sim Cx^{-\alpha}$, $x \rightarrow \infty$, $0 < \alpha < 2$, so it appears that the fatter the tail of the coefficient distribution, the more real zeros has the polynomial. But Zaporozhets's recent dissertation shows that if the coefficients have very long-tails and are symmetric, the expected number of real zeros is *bounded* in n . We show that the expected number of real zeros of the polynomial of degree n with real independent identically distributed coefficients with common characteristic function,

$$\phi(z) = e^{-A(\log|\frac{1}{z}|)^{-a}},$$

for $0 < |z| < 1$, and $\phi(0) = 1, \phi(z) \equiv 0$ for $1 \leq |z| < \infty$, with $1 < a$, and $A \geq a^{(a-1)}$, which has tails $P(\xi > x) \sim C(\log x)^{-a}$, as $x \rightarrow \infty$, is $\frac{a-1}{a-5} \log n$, asymptotically as $n \rightarrow \infty$. This shows the Zaporozhets phenomenon and also suggests the conjecture that the maximal number of real zeros, namely $1 \times \log n$, asymptotically, occurs when the common distribution has a tail faster than $C(\log x)^{-a}$ for any $a < \infty$, but slower than $x^{-\alpha}$ for any $\alpha > 0$.